Ex: Find the length of the polar curve

$r = \cos\theta$.

Sol: To complete one loop of $r=\cos\theta$, we need $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

$f(\theta) = \cos\theta$, $f'(\theta) = -\sin\theta$

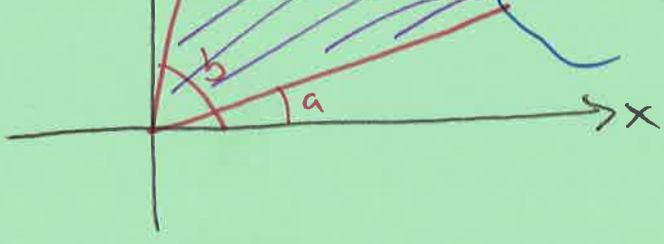
$$\Rightarrow L = \int_{-\pi/2}^{\pi/2} \sqrt{\cos^2\theta + (-\sin\theta)^2}\, d\theta = \int_{-\pi/2}^{\pi/2} \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta = \int_{-\pi/2}^{\pi/2} \sqrt{1}\, d\theta$$

$$= \theta \Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$$

Does this make sense?

Areas in Polar Coordinates

Let $f(\theta)$ be continuous on $a \le \theta \le b$. We would like to find the area swept out by the line segment from $(0,\theta)$ to $(f(\theta),\theta)$ as θ goes from a to b.

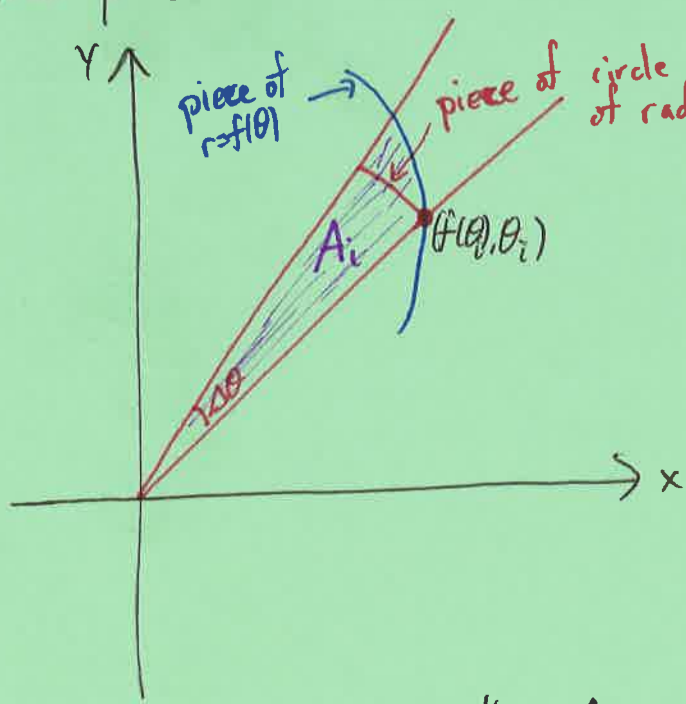


To avoid possible overlap, assume $b-a \leq 2\pi$ (however, sometimes we need to be more restrictive if $f(\theta)$ retraces itself sooner, e.g., $f(\theta) = \sin\theta$).

To find the area, let n be very large and partition the interval $a \leq \theta \leq b$:

$$a = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_i < \theta_{i+1} < \cdots < \theta_{n-1} < \theta_n = b.$$

where $\Delta\theta = \theta_{i+1} - \theta_i$. Let A_i be the area between θ_i and θ_{i+1}. In the section from θ_i to θ_{i+1}, draw in the piece of the circle of radius $f(\theta_i)$:



When $\Delta\theta$ is very small, $A_i \approx$ area of the sector of the circle.

Recall: The area of a sector of a circle with radius r and angle α is $A = \frac{1}{2}\alpha r^2$

So, $A_i \approx \frac{1}{2}[f(\theta_i)]^2 \Delta\theta$. Thus:

$$A = \lim_{n\to\infty} \sum_{i=0}^{n-1} A_i = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2}(f(\theta_i))^2 \Delta\theta = \int_a^b \frac{1}{2}(f(\theta))^2 d\theta$$

$$\boxed{A = \int_a^b \frac{1}{2}(f(\theta))^2 d\theta}$$

Ex: How much area does the spiral of Archimedes sweep out as θ goes from 0 to 2π?

Sol: $A = \int_0^{2\pi} \frac{1}{2}\theta^2 d\theta = \frac{1}{6}\theta^3 \Big|_0^{2\pi} = \frac{8\pi^3}{6} = \frac{4\pi^3}{3}$

Ex: Find the area enclosed by one petal of the rose

$$r = \sin 3\theta$$

Sol: One loop is given by $0 \le \theta \le \frac{\pi}{3}$, so:

$$A = \int_0^{\frac{\pi}{3}} \frac{1}{2}\sin^2 3\theta \, d\theta = \int_0^{\pi/3} \frac{1}{4}(1-\cos 6\theta)\, d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/3} = \left[\frac{1}{4}\left(\frac{\pi}{3} - 0\right)\right] - [0] = \frac{\pi}{12}$$

Equiangular Spirals

We've seen the spiral of Archimedes $r=f(\theta)=\theta$ before. Applying the derivative rule, we get

$$1=\theta\cdot\tan(\gamma-\tfrac{\pi}{2})$$

$$\Rightarrow \gamma=\arctan\left(\tfrac{1}{\theta}\right)+\tfrac{\pi}{2}$$

So, as θ increases, γ approaches $\frac{\pi}{2}$ (from above).

Here, γ is not a constant; but what do spirals with constant γ look like?

If γ is fixed, let $c=\tan(\gamma-\frac{\pi}{2})$, then

$$f'(\theta)=cf(\theta).$$

Thus $f(\theta)=Ae^{c\theta}$. Since $f(0)=Ae^{c\cdot 0}=A$, we can remove c & A to write

$$f(\theta)=f(0)e^{\tan(\gamma-\frac{\pi}{2})\theta}$$

as the general form of an equiangular spiral. If $\gamma>\frac{\pi}{2}$, the spiral expands, if $\gamma<\frac{\pi}{2}$, the spiral shrinks. (Of course, $\gamma=\frac{\pi}{2}$ is a circle!)

(Approximately)
Equiangular spirals occur frequently in nature!
For example in the shell of a nautilus, weather patterns, galaxy formations, etc.!

The Golden Ratio

Let a & b be two numbers and suppose $a > b > 0$.
a and b are said to be in the <u>golden ratio</u> if

$$\varphi = \frac{a}{b} = \frac{a+b}{a}$$

"Their ratio is equal to the ratio of the sum to the larger number."

φ is called the <u>golden ratio</u>. What is φ?

$$\varphi = \frac{a+b}{a} = \frac{a}{a} + \frac{b}{a} = 1 + \frac{1}{\varphi} \Rightarrow \varphi^2 = \varphi + 1$$

$$\Rightarrow \varphi^2 - \varphi - 1 = 0 \Rightarrow \varphi = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\varphi \text{ positive} \Rightarrow \boxed{\varphi = \frac{1+\sqrt{5}}{2}}$$

It is far from the case that every naturally occuring (approximately) equiangular spiral has angle φ, but it does happen... how can it happen?

Through the Fibonacci sequence!

(0,) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

We can write a recurrence relation:

$$a_0 = 0,\ a_1 = 1,\ a_{n+1} = a_n + a_{n-1} \quad (n \geq 1)$$

Observe what happens to the ratio of Fibonacci #'s:

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{a_n + a_{n-1}}{a_n} = \lim_{n\to\infty}\left(1 + \frac{a_{n-1}}{a_n}\right) = 1 + \frac{1}{L}$$

assume this exists

This is the same relation φ satisfies, so

$L = \varphi$ (b/c a_n is always positive)

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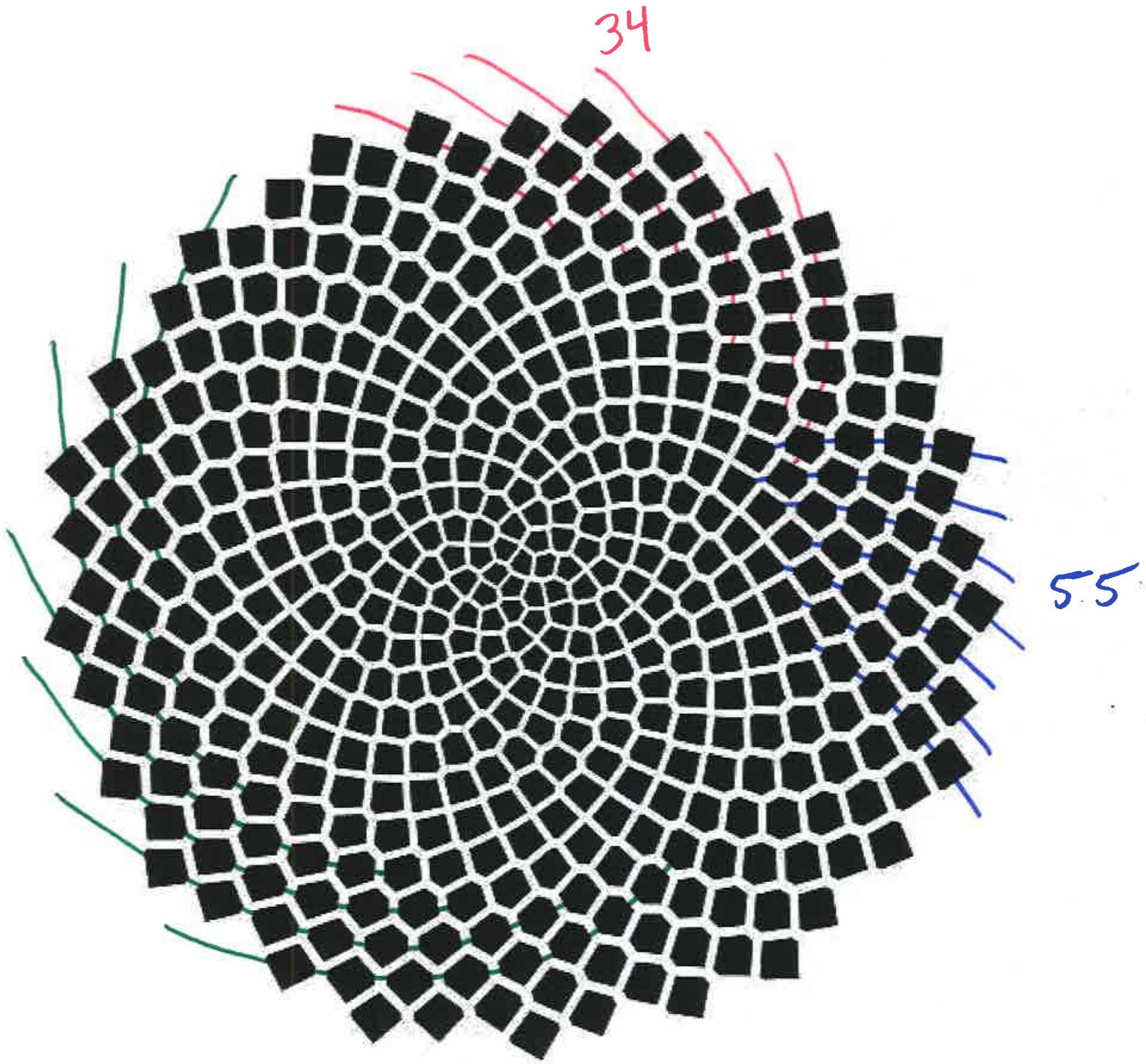

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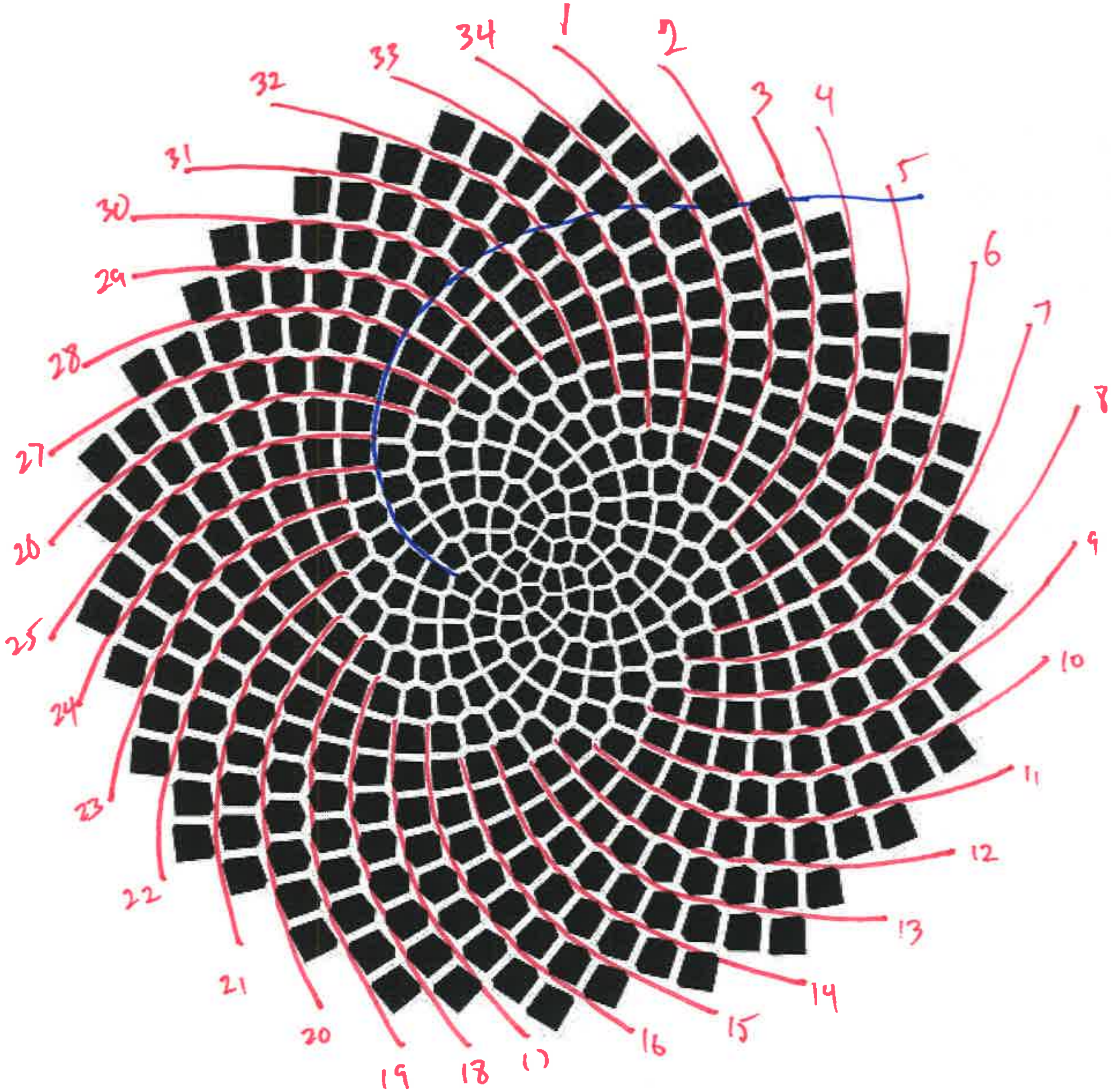
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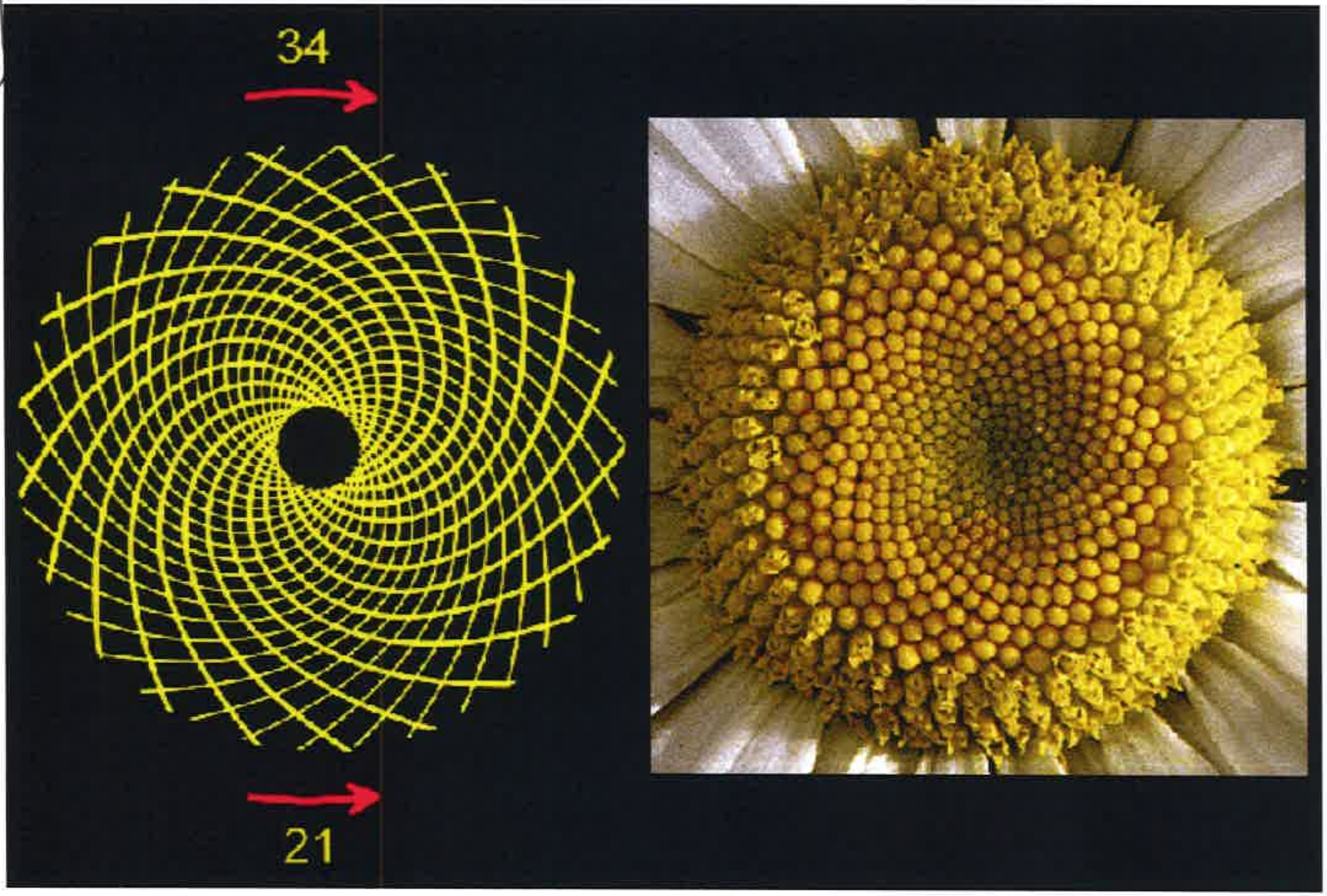
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